

# Self-Similar Random Processes and Infinite-Dimensional Configuration Spaces

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We discuss various infinite-dimensional configuration spaces that carry measures quasiinvariant under compactly-supported diffeomorphisms of a manifold  $M$  corresponding to a physical space. Such measures allow the construction of unitary representations of the diffeomorphism group, which are important to nonrelativistic quantum statistical physics and to the quantum theory of extended objects in  $M = \mathbb{R}^d$ . Special attention is given to measurable structure and topology underlying measures on generalized configuration spaces obtained from self-similar random processes (both for  $d = 1$  and  $d > 1$ ), which describe infinite point configurations having accumulation points.

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## I. INTRODUCTION

Let  $M$  be the manifold of physical space, usually taken to be  $d$ -dimensional Euclidean space  $\mathbb{R}^d$ . Let  $\text{Diff}^c(M)$  be the (infinite-dimensional) group of compactly-supported diffeomorphisms of  $M$ , under composition. The local current algebra approach to non-relativistic quantum mechanics led to the understanding that a wide variety of quantum systems could be described by constructing the continuous unitary representations (CURs) of  $\text{Diff}^c(M)$ , the group of compactly supported diffeomorphisms of  $M$  (under composition) [1, 2, 3, 4, 5, 6, 7].

To say that the diffeomorphism  $\phi$  of  $M$  has *compact support* means that for all points  $\mathbf{x} \in M$  that are outside some compact (and therefore bounded) region of  $M$ , the diffeomorphism acts as the identity operator:  $\phi(\mathbf{x}) \equiv \mathbf{x}$ . Our convention here will be to define the group product  $\phi_1\phi_2 = \phi_2 \circ \phi_1$ , where  $\circ$  denotes the composition of  $\phi_1, \phi_2 \in \text{Diff}^c(M)$ ; so that  $[\phi_1\phi_2](\mathbf{x}) = \phi_2(\phi_1(\mathbf{x}))$  for  $\mathbf{x} \in M$ . Thus we have a “right action” of the diffeomorphism group on the manifold.

In a very general framework, the Hilbert space where the unitary representation of  $\text{Diff}^c(M)$  can be realized is

the space of square-integrable functions,  $\mathcal{H} = L^2_\mu(\Delta, \mathcal{W})$ ; where  $\Delta$  is some *configuration space* on which the diffeomorphism group naturally acts (with a right action),  $\mu$  is a measure on  $\Delta$  satisfying appropriate technical conditions,  $\mathcal{W}$  is an inner product space, and the elements of  $\mathcal{H}$  are  $\mu$ -measurable functions  $\Psi(\gamma)$  on  $\Delta$  taking values in  $\mathcal{W}$ . The inner product of two such functions in  $\mathcal{H}$  is given by

$$(\Psi_1, \Psi_2) = \int_\Delta \langle \Psi_1(\gamma), \Psi_2(\gamma) \rangle_{\mathcal{W}} d\mu(\gamma) < \infty, \quad (1)$$

where  $\langle \Psi_1(\gamma), \Psi_2(\gamma) \rangle_{\mathcal{W}}$  denotes the inner product in  $\mathcal{W}$ . Then the operators  $V(\phi)$  defining a CUR are given by

$$[V(\phi)\Psi](\gamma) = \chi_\phi(\gamma)\Psi(\phi\gamma)\sqrt{\frac{d\mu_\phi(\gamma)}{d\mu}}; \quad (2)$$

where  $\phi\gamma$  refers to the action of the diffeomorphism  $\phi$  on  $\gamma \in \Delta$ , and where  $\chi_\phi : \mathcal{W} \rightarrow \mathcal{W}$  is a family of unitary operators acting in  $\mathcal{W}$  satisfying a certain cocycle equation (see below).

In this article we shall consider various candidates for a “large” configuration space, within which different choices of the space  $\Delta$  may be situated, that permit the construction of measures having the necessary property of quasiinvariance under diffeomorphisms. We then focus on the generalized configuration space  $\Sigma_M$  whose elements are *finite or countably infinite subsets* of  $M$ , and discuss ways of endowing it with a  $\sigma$ -algebra and a

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topology. The results underlie the construction of measures on generalized configuration spaces obtained from self-similar random processes in  $\mathbb{R}^d$  (both for  $d = 1$  and  $d > 1$ ), which describe infinite point configurations having accumulation points.

In Sec. II we briefly discuss the meaning of Eq. (2), reviewing the necessary concepts. Sec. III surveys some aspects of several possible choices of “large” configuration spaces, while Sec. IV focuses on topology and measurable structure in  $\Sigma_M$ . In Sec. V, we give a rapid overview of the construction of certain families of quasiinvariant measures in  $\Sigma_{\mathbb{R}^d}$  making use of self-similar random processes.

## II. MEASURES AND COCYCLES

The measure  $\mu$  that appears in Eq. (2) and in the definition of  $\mathcal{H}$  is, as usual, a countably-additive, positive real-valued function defined on a  $\sigma$ -algebra  $\mathcal{M}$  of subsets of  $\Delta$ . It is required to have the key property of *quasiinvariance* under the action of diffeomorphisms on  $\Delta$ .

In general, let  $G$  be a group of transformations of a measurable space  $(X, \mathcal{M})$ , where  $\mathcal{M}$  is a  $G$ -invariant  $\sigma$ -algebra of subsets of  $X$ . A measure  $\mu$  on  $\mathcal{M}$  is said to be *invariant* under  $G$  if and only if for all  $E \in \mathcal{M}$ , and for all  $g \in G$ ,  $\mu(gE) = \mu(E)$ . It is said to be *quasiinvariant* under  $G$  if and only if for all  $E \in \mathcal{M}$  such that  $\mu(E) > 0$ , and for all  $g \in G$ ,  $\mu(g(E)) > 0$ . That is,  $g \in G$  acts on  $X$  in such a way as to preserve the class of sets that have  $\mu$ -measure zero.

Quasiinvariance is *a fortiori* a consequence of invariance, but not conversely. For example, Lebesgue measure  $d\mathbf{x}$  on  $X = \mathbb{R}^d$  is invariant under the group of rigid motions (translations and rotations). It is quasiinvariant, but not invariant, under the group of compactly-supported diffeomorphisms of  $\mathbb{R}^d$ .

For  $\phi \in G = \text{Diff}^c(M)$  acting on  $X = \Delta$ , define the transformed measure  $\mu_\phi$  by setting  $\mu_\phi(E) = \mu(\phi(E))$  for any  $E \in \mathcal{M}$ . Because of the group structure and the  $G$ -invariance of  $\mathcal{M}$ , the quasiinvariance of  $\mu$  under  $G$  is equivalent to the absolute continuity of  $\mu_{\phi_1}$  with respect to  $\mu_{\phi_2}$  for any  $\phi_1, \phi_2 \in G$ . In particular, the quasiinvariance of  $\mu$  is necessary and sufficient for the existence of the Radon-Nikodym (RN) derivative  $(d\mu_\phi/d\mu)(\gamma)$  appearing in Eq. (2), for all elements  $\phi \in \text{Diff}^c(M)$ . For example, with  $M = \mathbb{R}^d$ ,  $\Delta = \mathbb{R}^d$ , and  $d\mu = d\mathbf{x}$ , we have  $(d\mu_\phi/d\mu)(\mathbf{x}) = \mathcal{J}_\phi(\mathbf{x})$ , the Jacobian of  $\phi$  at  $\mathbf{x}$ . Since  $\phi$  has compact support, we have  $\mathcal{J}_\phi(\mathbf{x}) \equiv 1$  outside some bounded region of  $\mathbb{R}^d$ .

The square root of the RN derivative in Eq. (2) is precisely the factor necessary to make the operators  $V(\phi)$  unitary in  $\mathcal{H}$ , since  $\chi_\phi(\gamma)$  is to be taken as acting unitarily in  $\mathcal{W}$  (see below). That is, the diffeomorphism  $\phi$  *moves* the argument of the wave function  $\Psi$ , and the square root factor *corrects* so that when we calculate the inner product  $(V(\phi)\Psi_1, V(\phi)\Psi_2)$  using Eq. (1), we find that we have merely made the change of variable  $\gamma' = \phi\gamma$  under the integral sign.

Let  $\mathcal{D}(M)$  be the space of real-valued, compactly-supported  $C^\infty$  functions  $f$  on  $M$ . We have then the natural semidirect product group  $\mathcal{D}(M) \times \text{Diff}^c(M)$ , with the group law given by

$$(f_1, \phi_1)(f_2, \phi_2) = (f_1 + f_2 \circ \phi_1, \phi_1\phi_2). \quad (3)$$

Now it may sometimes be the case that  $V(\phi)$  is a subrepresentation of a CUR of  $\mathcal{D}(M) \times \text{Diff}^c(M)$ , which we write  $U(f)V(\phi)$ . Then the operators  $U(f)$ ,  $f \in \mathcal{D}(M)$ , typically act in  $\mathcal{H}$  as multiplication operators, consistently with Eq. (2):

$$[U(f)\Psi](\gamma) = \exp[i\langle \gamma, f \rangle]\Psi(\gamma), \quad (4)$$

where  $\langle \gamma, f \rangle$  denotes an action of  $\gamma \in \Delta$  on  $f \in \mathcal{D}(M)$  as a continuous linear functional. That is, the configuration  $\gamma$  is here identified with a *distribution*, and  $\Delta$  is identified with a subset of the dual space  $\mathcal{D}'(M)$ . This is one of the possibilities discussed in Sec. III.

In Eq. (2),  $\chi_\phi : \mathcal{W} \rightarrow \mathcal{W}$  is a family of unitary operators in  $\mathcal{W}$  satisfying the *cocycle equation*

$$\chi_{\phi_1}(\gamma)\chi_{\phi_2}(\phi_1\gamma) = \chi_{\phi_1\phi_2}(\gamma), \quad (5)$$

which holds almost everywhere (a.e.) in  $\Delta$  for each pair of diffeomorphisms  $\phi_1, \phi_2$ . That is, Eq. (5) holds outside a  $\mu$ -measure zero set that in general may depend on  $\phi_1$  and  $\phi_2$ .

The cocycle equation follows directly from the condition that  $V$  respect the group law,  $V(\phi_1)V(\phi_2) = V(\phi_1\phi_2)$ . The trivial cocycle  $\chi_\phi(\gamma) \equiv I$  is always permitted, and in the case of a CUR describing  $N$  identical particles, this choice corresponds to Bose-Einstein statistics. Inequivalent choices of  $\chi_\phi$  (noncohomologous cocycles) are associated with Fermi-Dirac statistics, nontrivial phase effects, and anyon statistics in two space dimensions [8, 9, 10, 11, 12], as well as with certain nonlinear variations of quantum mechanics [13, 14, 15]. In the simplest cases,  $\mathcal{W}$  is just the 1-dimensional space of complex numbers  $\mathbb{C}$ , so that we have complex-valued wave functions on  $\Delta$ . Then the  $\chi_\phi$  act through multiplication by complex numbers of modulus 1. Higher-dimensional choices for  $\mathcal{W}$  are associated with paraparticles in  $\mathbb{R}^3$  and plektons in  $\mathbb{R}^2$  [16, 17, 18].

## III. GENERAL CONFIGURATION SPACES

To this point no universal configuration space for the representation theory of  $\text{Diff}^c(M)$  has been agreed upon. Consequently we have no one universal configuration space for the physics of systems with infinitely-many degrees of freedom in  $\mathbb{R}^d$ , within which specific choices of configuration spaces for particular systems are situated. This very likely reflects a gap in our present level of understanding. Let us describe here some choices that have been made, that allow the convenient description and interpretation of certain classes of unitary representations.

### A. Locally finite point configurations

The standard configuration space for statistical physics is the space  $\Gamma_M^{(\infty)}$  of countably infinite but locally finite subsets of  $M$ , where usually  $M = \mathbb{R}^d$ . Frequently one considers the disjoint union of this space with the spaces of  $N$ -point subsets; thus  $\Gamma_M = \bigsqcup_{N=1}^{\infty} \Gamma_M^{(N)} \bigsqcup \Gamma_M^{(\infty)}$  is the space of all locally finite subsets of  $M$ . Measures on the configuration space  $\Gamma_{\mathbb{R}^d}^{(\infty)}$  describe equilibrium states in  $\mathbb{R}^d$  in statistical mechanics; while  $\Gamma_{\mathbb{R}^d}^{(\infty)}$  also enters quantum theory in the description of infinite gases of quantum particles in  $\mathbb{R}^d$ .

Let  $|\gamma|$  denote the cardinality of the set  $\gamma$ . A configuration  $\gamma \subset \mathbb{R}^d$  in  $\Gamma_{\mathbb{R}^d}^{(\infty)}$  has the properties that  $|\gamma| = \aleph_0$ , while for any compact set  $K \subset \mathbb{R}^d$ ,  $|\gamma \cap K| < \infty$ . Then the diffeomorphism  $\phi \in \text{Diff}^c(\mathbb{R}^d)$  acts naturally on any configuration  $\gamma \in \Gamma_{\mathbb{R}^d}$  by its action on the individual elements of  $\gamma$ . This clearly respects the property of being finite or locally finite. Measures on  $\Gamma_{\mathbb{R}^d}^{(\infty)}$  that are quasiinvariant under diffeomorphisms have been extensively studied, and include Poisson measures and Gibbsian measures [3, 5, 6, 7, 19].

In particular, the choice of a Poisson measure  $d\mu^\sigma$  on  $\Gamma_{\mathbb{R}^d}^{(\infty)}$ , with intensity  $\sigma > 0$ , together with the trivial cocycle  $\chi_\phi \equiv 1$ , gives a CUR of  $\text{Diff}^c(\mathbb{R}^d)$  via Eq. (2). This representation describes the infinite, free quantum Bose gas having  $\sigma$  as its average particle number density [3]. Here we have, for any choice of  $\sigma$ ,

$$\frac{d\mu_\phi^\sigma}{d\mu^\sigma}(\gamma) = \prod_{\mathbf{x} \in \gamma} \mathcal{J}_\phi(\mathbf{x}). \quad (6)$$

Since  $\phi$  has compact support and  $\gamma$  is locally finite, it is evident that all but a finite number of terms in the infinite product of Jacobians in Eq. (6) are equal to 1. Thus this product gives a *finite, nonzero result* for the value of the RN derivative—expressing the fact that Poisson measures on  $\Gamma_{\mathbb{R}^d}^{(\infty)}$  are quasiinvariant under compactly-supported diffeomorphisms of  $\mathbb{R}^d$ .

### B. Configuration spaces of closed subsets

A much larger configuration space, introduced in early work by Ismagilov [20, 21, 22, 23], is the space  $\Omega_M$  of all (non-empty) closed subsets of the manifold  $M$ . For any closed set  $C \in \Omega_M$ , define the natural action of a diffeomorphism  $\phi \in \text{Diff}^c(M)$  on  $\Omega_M$  by  $\phi C = \{\phi(\mathbf{x}) | \mathbf{x} \in C\}$ . Evidently  $\phi C$  also belongs to  $\Omega_M$ , and we have a (right) group action.

A  $\sigma$ -algebra for  $\Omega_M$  is generated by the family of sets in  $\Omega_M$  consisting of all closed subsets of a given closed set. Thus for  $C \in \Omega_M$  (i.e., for  $C \subseteq M$  closed), let  $\Omega_C = \{C' \in \Omega_M | C' \subseteq C\}$ . Then let  $\mathcal{B}_{\Omega_M}$  be the smallest  $\sigma$ -algebra containing the family of sets  $\{\Omega_C\}_{C \subseteq M \text{ closed}}$ . This  $\sigma$ -algebra can also be obtained as the algebra of

Borel sets with respect to a topology on  $\Omega_M$ , for which a subbase is the family of sets  $\{C | C \cap \mathcal{O} \neq \emptyset\}_{\mathcal{O} \subseteq M \text{ open}}$ ; i.e., the family of subsets of  $\Omega_M$  whose elements *meet* a given open set  $\mathcal{O} \subseteq M$ .

Evidently any locally finite configuration  $\gamma \in \Gamma_M$  is also a closed subset of  $M$ , so that in general we have  $\Gamma_M \subset \Omega_M$ .

### C. Configuration spaces of generalized functions

Another possibility is to work with the dual space  $\mathcal{D}'(M)$ , as suggested by the CURs of the semidirect product group mentioned in Sec. I. That is, a configuration  $\gamma \in \mathcal{D}'(M)$  is a continuous, linear, real-valued functional on  $\mathcal{D}(M)$ —a *distribution* or *generalized function* on  $M$ . This is especially convenient for representing Eq. (4), as we can immediately write  $\langle \gamma, f \rangle$  for the value taken by  $\gamma$  on the function  $f \in \mathcal{D}(M)$ .

Diffeomorphisms act on  $\mathcal{D}'(M)$  by the dual to their action on  $\mathcal{D}(M)$ ; i.e.,  $\phi\gamma$  is defined for  $\gamma \in \mathcal{D}'(M)$  by  $\langle \phi\gamma, f \rangle = \langle \gamma, f \circ \phi \rangle$  for all  $f \in \mathcal{D}(M)$ . [With this definition and our earlier convention, we have  $(\phi_1\phi_2)\gamma = \phi_2(\phi_1\gamma)$ , so that the group action is a right action as desired.] A  $\sigma$ -algebra in  $\mathcal{D}'(M)$  may be built up directly from cylinder sets with Borel base [24], or  $\mathcal{D}'(M)$  can be endowed with the weak dual topology and measures constructed on the corresponding Borel  $\sigma$ -algebra.

Evidently  $\Gamma_M$ , or more specifically  $\Gamma_{\mathbb{R}^d}$ , may be identified naturally with a subset of  $\mathcal{D}'(M)$ , or  $\mathcal{D}'(\mathbb{R}^d)$ , by the correspondence

$$\gamma \rightarrow \sum_{\mathbf{x} \in \gamma} \delta_{\mathbf{x}}, \quad (7)$$

where  $\delta_{\mathbf{x}} \in \mathcal{D}'(M)$  is the evaluation functional (i.e., the Dirac  $\delta$ -function) defined by  $\langle \delta_{\mathbf{x}}, f \rangle = f(\mathbf{x})$ ,  $\mathbf{x} \in M$ .

The so-called “vague topology” in  $\Gamma_M$  is in fact the topology that  $\Gamma_M$  inherits from the weak dual topology in  $\mathcal{D}'(M)$ . While  $\Gamma_M$  is not a linear space, the larger space  $\mathcal{D}'(M)$  is. In addition to linear combinations of evaluation functionals (with possibly distinct real coefficients),  $\mathcal{D}'(M)$  contain other kinds of configurations of physical importance, that do not belong to  $\Gamma_M$  and in some cases are not easily identified with elements of  $\Omega_M$ . For example, configurations may include terms that are *derivatives* of  $\delta$ -functions, as well as generalized functions with support on embedded submanifolds of  $M$ .

### D. Configuration spaces of embeddings and immersions

Still another characterization of a “large” space of configurations in  $M$  begins with some other manifold (or manifold with boundary)  $L$ , together with a set of maps  $\alpha : L \rightarrow M$  that obey some specified regularity and continuity properties (for which there are numerous possible

choices). Then we call  $L$  the *parameter space* for the corresponding class of configurations, and  $M$  the *target space*. When  $\alpha$  is *injective* (so that self-intersection of the image of  $L$  in the target space is not permitted), we have a configuration space of *embeddings*, while without any such restriction we have a larger space of *immersions*.

We have at the outset the choice of considering *parameterized* or *unparameterized* maps. A space of parameterized  $C^k$  immersions consists of mappings  $\alpha(\theta)$ ,  $\theta \in L$ , that are  $C^k$  for some fixed integer  $k \geq 0$ . For  $\phi \in \text{Diff}^c(M)$ , the formula  $[\phi\alpha](\theta) = \phi(\alpha(\theta))$  (i.e.,  $\phi\alpha = \phi \circ \alpha$ ) defines the desired (right) group action on the space of parameterized immersions. In addition, the group  $\text{Diff}(L)$  acts on the space of immersions (as a left action) by *reparameterization*, so that for  $\psi \in \text{Diff}(L)$ ,  $\psi : \alpha \rightarrow \alpha \circ \psi$ .

Then an unparameterized immersion is just the *image set*  $K = \alpha(L) \subset M$ , where the parameterization of  $K$  has been disregarded. Alternatively, we can think of the unparameterized immersion as an equivalence class of parameterized immersions *modulo* reparameterization. Note that the action of  $\text{Diff}^c(M)$  on the space of (parameterized or unparameterized) immersions leaves the corresponding space of embeddings invariant as a subset, and preserves the continuity properties of configurations in the space.

If  $L$  is the circle  $S^1$ , for instance, configurations are  $C^k$  loops in  $M$ . The embeddings are the non-self-intersecting loops. The action of the diffeomorphism group also respects the knot class of the loop. If  $L$  is the closed interval  $[0, 2\pi]$ , configurations are finite arcs in  $M$ . Further possibilities include ribbons, tubes, or higher-dimensional submanifolds of  $M$ .

The configuration space of unparameterized immersions of  $L$  in  $M$  is a subset of the configuration space  $\Omega_M$ , invariant (as a set) under the action of  $\text{Diff}^c(M)$ . This description thus allows us to *refine*  $\Omega_M$  as sensitively as desired, according to the topological and continuity properties of extended configurations.

For example, quantized vortex configurations in ideal, incompressible fluids are obtained from representations of groups of (area- and volume-preserving) diffeomorphisms of  $\mathbb{R}^2$  and  $\mathbb{R}^3$ . For planar fluids, pure point vortices are not permitted quantum-mechanically, but one-dimensional *filaments* of vorticity are allowed. Similarly, in  $\mathbb{R}^3$  pure filaments are kinematically forbidden, while two-dimensional vortex *surfaces*, e.g. ribbons or tubes, can occur [25, 26, 27, 28]. But a major gap is the construction of measures, quasiinvariant under diffeomorphisms, directly on spaces of filaments or tubes. One approach to the filament case has been suggested by Shavgulidze [29].

Naturally a nonrelativistic quantum theory of strings, with  $\mathbb{R}^d$  as the target space, also depends on quasiinvariant measures on the space of loops.

In addition, we remark that diffeomorphism-invariant measures are important to the long-standing problem of finding consistent theories for quantized gravity; for

instance, Ashtekar and Lewandowski have constructed a faithful, diffeomorphism-invariant measure on a compactification of the space of gauge-equivalent connections [30, 31].

Reparameterization invariance has nice consequences for quantum mechanics, when expressed in terms of diffeomorphism group representations. Note in particular that we can consider the  $N$ -particle configuration space  $\Gamma_M^{(N)}$  as a special case of embeddings *modulo* reparameterization, with the discrete manifold  $L = \{1, \dots, N\}$ . The group  $\text{Diff}(L)$  in this case is the symmetric group  $S_N$ . The corresponding configuration space of parameterized embeddings is the space of *ordered*  $N$ -tuples  $(\mathbf{x}_1, \dots, \mathbf{x}_N)$  of distinct points,  $\mathbf{x}_j \neq \mathbf{x}_k$  for  $j \neq k$ . The space of parameterized immersions of  $L$  in  $M$  includes the  $N$ -tuples with coincident points.

### E. The configuration space of countable subsets of $\mathbb{R}^d$

The idea pursued in the balance of this article is the construction of measures, quasiinvariant under diffeomorphisms of  $\mathbb{R}^d$ , on the space  $\Sigma_{\mathbb{R}^d}^{(\infty)}$  of countably infinite subsets of the physical space  $\mathbb{R}^d$  that are *not* necessarily locally finite. Alternatively, we may work on the space  $\Sigma_{\mathbb{R}^d}$  whose elements are subsets  $\gamma \subset \mathbb{R}^d$  that are finite or countably infinite, with

$$\Sigma_{\mathbb{R}^d} = \bigsqcup_{N=1}^{\infty} \Gamma_{\mathbb{R}^d}^{(N)} \bigsqcup \Sigma_{\mathbb{R}^d}^{(\infty)}. \quad (8)$$

We call this the space of *generalized configurations*.

Our main mathematical motivation for working with this space is that measures on it can be constructed by means of random point processes on spaces of *infinite sequences* of points in  $\mathbb{R}^d$ . We shall project the measure  $\mu$  on  $[\mathbb{R}^d]^\infty$  that results from such a point process to define the corresponding measure  $\hat{\mu}$  on  $\Sigma_{\mathbb{R}^d}$ , thus obtaining a measure on the space  $\Sigma_{\mathbb{R}^d}^{(\infty)}$ .

A physical motivation for this direction of work is the goal of constructing quasiinvariant measures for spatially-extended systems, which is in general an unsolved problem. Since  $\mathbb{R}^d$  is separable, any closed set in  $\mathbb{R}^d$  can be obtained as the closure of an element of  $\Sigma_{\mathbb{R}^d}$ ; so that the *closure map*  $\gamma \rightarrow \bar{\gamma}$  from  $\Sigma_{\mathbb{R}^d}$  to  $\Omega_{\mathbb{R}^d}$  is surjective. Thus our present approach—which puts us into a still larger configuration space than that of Ismagilov—may permit point-like approximations to embedded manifold configurations.

Apart from this general consideration, the specific measures we can construct appear to have a direct interpretation as descriptive of idealized quantum or statistical configurations forming “particle clouds” about a locus of condensation. These allow for a kind of “phase transition” from a rarefied to a condensed phase, as the self-similarity parameter passes through a critical value.

Let us write  $\omega = (\mathbf{x}_j) \in [\mathbb{R}^d]^\infty$  to denote an infinite sequence, with  $j = 1, 2, 3, \dots$

Now generalized configurations, like infinite sequences, can have accumulation points. A point  $\mathbf{x} \in \mathbb{R}^d$  is an accumulation point of a set  $\gamma \subset \mathbb{R}^d$ —or, respectively, of an infinite sequence  $\omega = (\mathbf{x}_j) \in [\mathbb{R}^d]^{(\infty)}$ —if for any neighborhood  $\mathcal{U}$  of  $\mathbf{x}$ , the set  $\mathcal{U} - \{\mathbf{x}\}$  contains infinitely many points of  $\gamma$  (respectively,  $\omega$ ). An accumulation point of  $\gamma$  may or may not itself be an element of  $\gamma$ . Evidently diffeomorphisms of  $\mathbb{R}^d$  act naturally on generalized configurations, respecting accumulation points: if  $\mathbf{x} \in \mathbb{R}^d$  is an accumulation point of  $\gamma \in \Sigma_{\mathbb{R}^d}^{(\infty)}$ , then  $\phi(\mathbf{x})$  is an accumulation point of  $\phi\gamma$ . The points belonging to configurations in  $\Sigma_{\mathbb{R}^d}^{(\infty)}$  can cluster in such a manner as to yield fractals or even more complicated objects.

The set of sequences containing coincident points is called the "diagonal"  $D$  in  $[\mathbb{R}^d]^\infty$ ; that is,  $D = \{(\mathbf{x}_j) \in [\mathbb{R}^d]^\infty \mid \mathbf{x}_k = \mathbf{x}_\ell \text{ (for some } k \neq \ell)\}$ . Typically  $D$  is of measure zero for the point processes of interest, and for technical reasons it will often be convenient to exclude it. We have the natural projection from the sequence space to the configuration space,  $p : [\mathbb{R}^d]^\infty \rightarrow \Sigma_{\mathbb{R}^d}$ , given by  $p[(\mathbf{x}_j)] = \{\mathbf{x}_j\}$ . The image of  $[\mathbb{R}^d]^\infty$  under  $p$  is *all* of  $\Sigma_{\mathbb{R}^d}$ , since the possibility of repeated entries in elements of  $[\mathbb{R}^d]^\infty$  permits the corresponding configurations to be finite as well as infinite. Then  $[\mathbb{R}^d]^\infty$  can also be thought of as a fiber space over  $\Sigma_{\mathbb{R}^d}$ . It is natural to consider also the restriction of  $p$  to sequences without repeated entries,  $p : [\mathbb{R}^d]^\infty - D \rightarrow \Sigma_{\mathbb{R}^d}^{(\infty)}$  (which is surjective).

Note that the space  $\Sigma_{\mathbb{R}^d}^{(\infty)}$  may also be regarded as a special case of the space of unparameterized embeddings discussed in the preceding subsection. The target space  $M$  is  $\mathbb{R}^d$ ; the parameter space  $L$  is  $\mathbb{N}$  (the set of natural numbers); and  $\text{Diff}(L)$  is the group  $S^\infty$  of bijections of  $\mathbb{N}$ . Of course,  $[\mathbb{R}^d]^\infty - D$  is then seen as the space of parameterized embeddings of  $L$  into  $M$ ; while  $[\mathbb{R}^d]^\infty$  itself is the space of parameterized immersions.

For any diffeomorphism  $\phi$  of  $\mathbb{R}^d$ , we have  $\phi p[(\mathbf{x}_j)] = \{\phi(\mathbf{x}_j)\} = p[(\phi(\mathbf{x}_j))]$ . Thus we can project a probability measure on the sequence-space  $[\mathbb{R}^d]^\infty$  or  $[\mathbb{R}^d]^\infty - D$ , constructed as is usual from an infinite sequence of conditional probability densities on  $\mathbb{R}^d$ , to a probability measure on the configuration space  $\Sigma_{\mathbb{R}^d}^{(\infty)}$ , consistent with the action of  $\text{Diff}^c(\mathbb{R}^d)$ .

In earlier work, it was shown how for the one-dimensional manifolds  $\mathbb{R}^1$  or  $S^1$ , *self-similar* point processes in the manifold lead quite generally through such a construction to quasiinvariant measures on the configuration space of countably infinite subsets [32, 33, 34, 35, 36]. The quasiinvariance is intimately related to the self-similarity. In Sec. IV we shall discuss further the relevant  $\sigma$ -algebra on this configuration space, which lays the foundation for completing the rigorous proofs of earlier conjectures. Then we shall indicate how the generalization to  $d > 1$  is carried out [37].

#### IV. TOPOLOGY AND MEASURABLE STRUCTURE ON $\Sigma_{\mathbb{R}^d}$

There are at least two possible approaches to defining a  $\sigma$ -algebra on the generalized configuration space  $\Sigma_{\mathbb{R}^d}^{(\infty)}$ .

##### A. Indirect approach through $[\mathbb{R}^d]^\infty$

The indirect approach makes use of the sequence space  $[\mathbb{R}^d]^\infty$ , which is endowed with the well-known *weak product topology*  $\tau_w$ . Let us write  $\mathbf{x}_j(\omega)$  for the  $j$ th entry of  $\omega \in [\mathbb{R}^d]^\infty$ . The weak topology is then the coarsest topology for which all the natural projections  $\pi_j : [\mathbb{R}^d]^\infty \rightarrow \mathbb{R}^d$  given by  $\omega \rightarrow \mathbf{x}_j(\omega)$  are continuous. This topology is inherited by  $[\mathbb{R}^d]^\infty - D$ .

Let  $\mathcal{B}([\mathbb{R}^d]^\infty)$  denote the  $\sigma$ -algebra of Borel sets in  $[\mathbb{R}^d]^\infty$  with respect to  $\tau_w$ . This naturally induces a  $\sigma$ -algebra in  $\Sigma_{\mathbb{R}^d}$ —namely, the largest  $\sigma$ -algebra with the property that the projection  $p : [\mathbb{R}^d]^\infty \rightarrow \Sigma_{\mathbb{R}^d}$  is measurable [33, 34]. More precisely, we introduce in  $\Sigma_{\mathbb{R}^d}$  the  $\sigma$ -algebra

$$\mathcal{P}_w(\Sigma_{\mathbb{R}^d}) := \{A \subseteq \Sigma_{\mathbb{R}^d} \mid p^{-1}(A) \in \mathcal{B}([\mathbb{R}^d]^\infty)\}, \quad (9)$$

to which each of the subsets  $\Gamma_{\mathbb{R}^d}^{(N)}$ ,  $N = 1, 2, 3, \dots$ , as well as  $\Sigma_{\mathbb{R}^d}^{(\infty)}$ , belongs.

Evidently the set of accumulation points of an infinite sequence in  $\mathbb{R}^d$  or  $\mathbb{R}^d - D$  may be empty, finite and non-empty, countably infinite, or uncountably infinite. Since accumulation points in  $\mathbb{R}^d$  depend only on the *set*  $\gamma = \{\mathbf{x}_j\}$ , and not specifically on the *sequence*  $(\mathbf{x}_j)$ , all the distinct elements of  $p^{-1}(\gamma)$  have precisely the same accumulation points.

Now it is straightforward to demonstrate that various sets of interest in  $\Sigma_{\mathbb{R}^d}^{(\infty)}$  belong to  $\mathcal{P}_w$ , by showing that the corresponding sets of sequences belong to  $\mathcal{B}([\mathbb{R}^d]^\infty - D)$ . A series of lemmas in earlier work [38, 39] shows that the set  $[\mathbb{R}^d]^\infty - D$  itself belongs to  $\mathcal{B}([\mathbb{R}^d]^\infty)$ , and that the following subsets of  $[\mathbb{R}^d]^\infty - D$  are likewise Borel: the set of all nonrepeating sequences having precisely  $n$  elements in a given compact set  $K \subset \mathbb{R}^d$ ; the set of all locally finite nonrepeating sequences; and the set of all nonrepeating sequences having precisely  $N$  accumulation points in  $K$ . Each of these sets is the inverse image in  $[\mathbb{R}^d]^\infty - D$  (under the projection  $p$ ) of a set in  $\Sigma_{\mathbb{R}^d}^{(\infty)}$ ; hence the corresponding sets in  $\Sigma_{\mathbb{R}^d}^{(\infty)}$  are measurable.

In fact,  $\mathcal{P}_w(\Sigma_{\mathbb{R}^d}^{(\infty)})$  is sufficiently rich to permit us to count the numbers of accumulation points of configurations that are located in arbitrary Borel sets of  $\mathbb{R}^d$  (not just compact sets). In particular, the subsets  $\Sigma_{\mathbb{R}^d, N}^{(\infty)} \subset \Sigma_{\mathbb{R}^d}^{(\infty)}$  consisting of generalized configurations having exactly  $N$  accumulation points in  $\mathbb{R}^d$  are measurable. The inverse image  $p^{-1}(\Sigma_{\mathbb{R}^d, N}^{(\infty)})$  is the set of infinite sequences having precisely  $N$  accumulation points, which we denote by  $[\mathbb{R}^d]^\infty_N \subset [\mathbb{R}^d]^\infty$  (for  $N = 0, 1, 2, \dots$ ).

Suppose that we have a probability measure  $\mu$  on  $[\mathbb{R}^d]^\infty$  or  $[\mathbb{R}^d]^\infty - D$ . Then we obtain a probability measure  $\hat{\mu}$  on  $\Sigma_{\mathbb{R}^d}$  by defining, for all  $A \in \mathcal{P}_w(\Sigma_{\mathbb{R}^d})$ ,  $\hat{\mu}(A) = \mu(p^{-1}(A))$ . The most straightforward way to construct a countably additive measure  $\mu$  on  $[\mathbb{R}^d]^\infty$  [with the  $\sigma$ -algebra  $\mathcal{B}([\mathbb{R}^d]^\infty)$ ] is to specify a compatible family of measures on the finite-dimensional spaces from which  $[\mathbb{R}^d]^\infty$  is constructed as the projective limit. The existence of the corresponding measure  $\mu$  is then assured by Kolmogorov's theorem. If  $\mu$  is quasiinvariant under diffeomorphisms of  $\mathbb{R}^d$ , then our construction ensures that  $\hat{\mu}$  is also quasiinvariant as desired.

### B. Direct approach

The more direct approach to constructing a  $\sigma$ -algebra on  $\Sigma_{\mathbb{R}^d}^{(\infty)}$  is simply to specify a generating set of subsets of  $\Sigma_{\mathbb{R}^d}$  or  $\Sigma_{\mathbb{R}^d}^{(\infty)}$  for the  $\sigma$ -algebra, or else to introduce a topology in  $\Sigma_{\mathbb{R}^d}$  or  $\Sigma_{\mathbb{R}^d}^{(\infty)}$  and to take as our  $\sigma$ -algebra the Borel sets with respect to that topology.

For instance, we may begin with Ismagilov's  $\sigma$ -algebra on  $\Omega_{\mathbb{R}^d}$  described above, and lift it to a  $\sigma$ -algebra  $\mathcal{I}(\Sigma_{\mathbb{R}^d})$  using the closure map. The generating family for  $\mathcal{I}(\Sigma_{\mathbb{R}^d})$  becomes all sets of the form  $\{\gamma \in \Sigma_{\mathbb{R}^d} \mid \gamma \subseteq F\}$ , where  $F \in \Omega_{\mathbb{R}^d}$  is closed. Because  $F$  is closed,  $\gamma \subseteq F$  if and only if  $\bar{\gamma} \subseteq F$ . The complement of a set in this generating family is the set  $\mathcal{O}_U = \{\gamma \in \Sigma_{\mathbb{R}^d} \mid \gamma \cap U \neq \emptyset\}$ , the set of all configurations that *meet* the open set  $U \subseteq \mathbb{R}^d$ , where  $U$  is  $\mathbb{R}^d - F$ . The collection of sets  $\{\mathcal{O}_U \mid U \subseteq \mathbb{R}^d \text{ open}\}$  likewise serves as generating family for  $\mathcal{I}(\Sigma_{\mathbb{R}^d})$  [37]. The subsets  $\Gamma_{\mathbb{R}^d}^{(N)}$  and  $\Sigma_{\mathbb{R}^d}^{(\infty)}$  of  $\Sigma_{\mathbb{R}^d}$  belong to  $\mathcal{I}(\Sigma_{\mathbb{R}^d})$ .

We can make use of these families of sets to introduce a natural topology on  $\Sigma_{\mathbb{R}^d}$ . Define a subbase of open sets for a topology  $\tau_o$  in  $\Sigma_{\mathbb{R}^d}$  to be  $\{\mathcal{O}_U \mid U \subseteq \mathbb{R}^d \text{ open}\}$ . Note that for any index set  $I$ ,  $\cup_{\alpha \in I} \mathcal{O}_{U_\alpha} = \mathcal{O}_{[\cup_{\alpha \in I} U_\alpha]}$ , while  $\cap_{j=1, \dots, n} \mathcal{O}_{U_j} \supset \mathcal{O}_{[\cap_{j=1, \dots, n} U_j]}$ . The finite intersections of sets in the subbase form a base for  $\tau_o$ .

In the topology  $\tau_o$ , the subsets  $\Gamma_{\mathbb{R}^d}^{(n)} \subset \Sigma_{\mathbb{R}^d}$  (for  $n > 1$ ) and  $\Sigma_{\mathbb{R}^d}^{(\infty)} \subset \Sigma_{\mathbb{R}^d}$  are neither open nor closed. However, for each  $N \geq 0$ ,  $\{\gamma \mid |\gamma| \leq N\} = \bigsqcup_{n=1}^N \Gamma_{\mathbb{R}^d}^{(n)}$  is closed. Of course, we may also consider separately the topology induced in  $\Sigma_{\mathbb{R}^d}^{(\infty)}$  by  $\tau_o$ .

Now the  $\sigma$ -algebra  $\mathcal{I}(\Sigma_{\mathbb{R}^d})$ , that we obtained by lifting Ismagilov's  $\sigma$ -algebra to  $\Sigma_{\mathbb{R}^d}$  by the inverse image of the closure map, is precisely the Borel  $\sigma$ -algebra  $\mathcal{B}_o(\Sigma_{\mathbb{R}^d})$  with respect to the topology  $\tau_o$ . Indeed, we noted already that the complement of  $\mathcal{O}_U$  in  $\Sigma_{\mathbb{R}^d}$  is just  $\{\gamma \in \Sigma_{\mathbb{R}^d} \mid \gamma \subseteq \mathbb{R}^d - U\}$ . Thus we have immediately that  $\mathcal{B}_o(\Sigma_{\mathbb{R}^d})$  contains  $\mathcal{I}(\Sigma_{\mathbb{R}^d})$ , and the closure map is  $\tau_o$ -Borel measurable with respect to Ismagilov's  $\sigma$ -algebra on  $\Omega_{\mathbb{R}^d}$ . Conversely, let  $\{U_j \mid j = 1, 2, 3, \dots\}$  be a countable base for the topology in  $\mathbb{R}^d$ . Then  $\{\mathcal{O}_{U_j}\}$  is a countable subbase for  $\tau_o$ , and the finite intersections of such sets form a countable base for  $\tau_o$  whose elements are obtained directly from the generating family for  $\mathcal{I}(\Sigma_{\mathbb{R}^d})$ . Hence  $\mathcal{B}_o(\Sigma_{\mathbb{R}^d}) = \mathcal{I}(\Sigma_{\mathbb{R}^d})$ .

Sakuraba constructs and discusses a related topology  $\tau_s$  on  $\Sigma_M$  (here  $M = \mathbb{R}^d$ ), obtained as a quotient of the product topology on the disjoint union of  $M^n$ ,  $n = 1, 2, 3, \dots$ , and  $M^\infty$  with respect to the symmetric groups  $S_n$  and the infinite symmetric group [37]. In this construction, the topology on  $\Sigma_M$  is the sum of topologies on the components  $\Gamma_M^{(n)}$  and  $\Sigma_M^{(\infty)}$ ; and each of the subsets  $\Gamma_M^{(n)}$  is *both* closed and open. Restricted to each component,  $\tau_s$  coincides with the topology induced by  $\tau_o$ . Thus the family of Borel sets of  $\tau_s$  coincides with the family of Borel sets of  $\tau_o$ .

The fact that  $\mathcal{I}(\Sigma_{\mathbb{R}^d}) \subset \mathcal{P}_w(\Sigma_{\mathbb{R}^d})$  is straightforward: since

$$p^{-1}(\mathcal{O}_U) = \bigcup_{j=1}^{\infty} \{\omega \in [\mathbb{R}^d]^\infty \mid \mathbf{x}_j(\omega) \in U\}, \quad (10)$$

the inverse image of  $\mathcal{O}_U$  is open in the weak topology of  $[\mathbb{R}^d]^\infty$ , and therefore  $\mathcal{O}_U$  belongs to  $\mathcal{P}_w(\Sigma_{\mathbb{R}^d})$ . But  $\mathcal{I}(\Sigma_{\mathbb{R}^d})$  is in fact smaller than  $\mathcal{P}_w(\Sigma_{\mathbb{R}^d})$ , and too small for certain purposes. Indeed, by our previous result any  $\tau_o$ -Borel set  $B$  is the inverse image under the closure map of a set in the  $\sigma$ -algebra on  $\Omega_{\mathbb{R}^d}$  generated by the sets  $\Omega_F$ ; thus it has the property that if  $\gamma \in B$ ,  $\bar{\gamma} \in B$ .

But it is easy to construct sets in  $\mathcal{P}_w(\Sigma_{\mathbb{R}^d})$  that do not have this property. For example, define the set  $\mathcal{O}^V$  of all configurations  $\gamma \in \Sigma_{\mathbb{R}^d}$  that are subsets of a given open set  $V$ . Evidently, there exist countably infinite subsets of  $V$  whose closures are no longer subsets of  $V$ , so  $\mathcal{O}^V$  does not belong to  $\mathcal{I}(\Sigma_{\mathbb{R}^d})$ . However  $\mathcal{O}^V$  does belong to  $\mathcal{P}_w(\Sigma_{\mathbb{R}^d})$ , which follows from the fact that

$$p^{-1}(\mathcal{O}^V) = p^{-1}(\{\gamma \in \Sigma_{\mathbb{R}^d} \mid \gamma \subset V\})$$

$$= \bigcap_{j=1}^{\infty} \{\omega \mid \mathbf{x}_j(\omega) \in V\}. \quad (11)$$

Thus  $\mathcal{I}(\Sigma_{\mathbb{R}^d}^{(\infty)}) \neq \mathcal{P}_w(\Sigma_{\mathbb{R}^d}^{(\infty)})$ . The  $\sigma$ -algebra  $\mathcal{I}(\Sigma_{\mathbb{R}^d})$  is just not large enough for us to be able to count the number of points in a configuration that belong to a given open set in  $\mathbb{R}^d$ .

This example suggests consideration of the *Vietoris topology* on subsets of  $\mathbb{R}^d$ , restricted to  $\Sigma_{\mathbb{R}^d}$  or to  $\Sigma_{\mathbb{R}^d}^{(\infty)}$ . Let us call this topology  $\tau_v$ . A subbase for  $\tau_v$  is given by sets of the form  $\mathcal{O}^V \cap \mathcal{O}_U$  where  $U$  and  $V$  are open; so that  $\mathcal{O}^V$  is itself open in  $\tau_v$ . The Vietoris topology has many nice properties [40, 41, 42]. Considering then the  $\sigma$ -algebra  $\mathcal{B}_v(\Sigma_{\mathbb{R}^d})$  of Borel sets with respect to  $\tau_v$ , we have  $\mathcal{I}(\Sigma_{\mathbb{R}^d}) \subset \mathcal{B}_v(\Sigma_{\mathbb{R}^d})$ , but  $\mathcal{I}(\Sigma_{\mathbb{R}^d}) \neq \mathcal{B}_v(\Sigma_{\mathbb{R}^d})$ . Furthermore,  $\mathcal{B}_v(\Sigma_{\mathbb{R}^d}) \subset \mathcal{P}_w(\Sigma_{\mathbb{R}^d})$ . To show this, consider again a countable base  $\{U_j, j = 1, 2, 3, \dots\}$  for the topology in  $\mathbb{R}^d$ . A countable subbase for  $\tau_v$  is then  $\{\mathcal{O}^{U_j} \cap \mathcal{O}_{U_k}, j, k = 1, 2, 3, \dots\}$ ; and a countable base for  $\tau_v$  consists of finite intersections of such sets. Since  $p^{-1}(\mathcal{O}^{U_j})$  and  $p^{-1}(\mathcal{O}_{U_k})$  are both Borel in  $[\mathbb{R}^d]^\infty$ , the inverse image of any open set in  $\tau_v$  is Borel in  $[\mathbb{R}^d]^\infty$ , which suffices for the result.

We have not, however, determined whether  $\mathcal{B}_v(\Sigma_{\mathbb{R}^d})$  is or is not strictly smaller than  $\mathcal{P}_w(\Sigma_{\mathbb{R}^d})$ .

## V. SELF-SIMILAR RANDOM POINT PROCESSES IN $\mathbb{R}^d$ AND QUASIINVARIANT MEASURES

Now we are prepared to construct measures on the  $\sigma$ -algebra  $\mathcal{B}([\mathbb{R}^d]^\infty)$  by means of random point processes, using sequences of conditional probability densities. When we do so, it turns out that the RN derivatives under transformations by diffeomorphisms take the form of an infinite product,

$$\frac{d\mu_\phi}{d\mu}(\omega) = \prod_{j=1}^{\infty} u_{j,\phi}(\omega). \quad (12)$$

Here  $\omega \in [\mathbb{R}^d]^\infty$ , and the  $u_{j,\phi}(\omega)$  are measurable functions that depend only on the first  $j$  entries of  $\omega$ .

Quasiinvariance of  $\mu$  then requires that (12) converge to a non-zero, non-infinite limit almost everywhere in  $\mu$ , for each  $\phi$ . This means that the individual terms  $u_{j,\phi}(\omega)$  must approach 1 sufficiently rapidly, as  $j \rightarrow \infty$ . Under conditions that in fact hold for the measures discussed here, these convergence properties have also been proven sufficient to ensure the quasiinvariance of  $\mu$  [37], and as a direct consequence, the quasiinvariance of the projected measure  $\hat{\mu}$  on  $\Sigma_{\mathbb{R}^d}^{(\infty)}$ .

Let  $f(\mathbf{x}_j | \mathbf{x}_1, \dots, \mathbf{x}_{j-1})$  be a non-singular probability density on  $\mathbb{R}^d$  for selection of the point  $\mathbf{x}_j$ , conditioned on the previously-selected points  $\mathbf{x}_1, \dots, \mathbf{x}_{j-1}$  in some random sequence. Then  $d\mu_j(\mathbf{x}_j) = f(\mathbf{x}_j | \mathbf{x}_1, \dots, \mathbf{x}_{j-1}) d\mathbf{x}_j$  defines a conditional (Borel) probability measure  $\mu_j$  on  $\mathbb{R}^d$  that depends measurably on the  $j-1$  real parameters  $\mathbf{x}_1, \dots, \mathbf{x}_{j-1}$  (the positions of the first  $j-1$  particle coordinates), and is absolutely continuous with respect to the Lebesgue measure  $d\mathbf{x}_j$ . We can interpret the joint probability measure for the first  $k$  points, specified by  $d\mu^{(k)} = \prod_{j=1}^k d\mu_j$ , as a measure on  $[\mathbb{R}^d]^\infty$ ; and the sequence  $(\mu^{(k)})$ ,  $k = 1, 2, 3, \dots$ , is then a compatible family of probability measures.

By Kolmogorov's theorem, there is a unique measure  $\mu$  on  $[\mathbb{R}^d]^\infty$  determined by the sequence  $(\mu^{(k)})$ . Under transformation by  $\phi \in \text{Diff}^c(\mathbb{R}^d)$ , the RN derivative for  $\mu^{(k)}$  (when it exists) is given by the finite product

$$\frac{d\mu_\phi^{(k)}}{d\mu^{(k)}}(\omega) = \prod_{j=1}^k \frac{d\mu_{j,\phi}}{d\mu_j}(\omega), \quad (13)$$

where

$$\frac{d\mu_{j,\phi}}{d\mu_j}(\omega) = \frac{f(\phi(\mathbf{x}_j) | \phi(\mathbf{x}_1), \dots, \phi(\mathbf{x}_{j-1}))}{f(\mathbf{x}_j | \mathbf{x}_1, \dots, \mathbf{x}_{j-1})} \mathcal{J}_\phi(\mathbf{x}_j). \quad (14)$$

The quasiinvariance of  $\mu^{(k)}$  is assured as long as the RN derivative in Eq. (13) is almost everywhere positive and

finite. Now, as anticipated, in the infinite-dimensional case quasiinvariance of the measure  $\mu$  under diffeomorphisms turns out to depend on the behavior of the infinite product in Eq. (12), with  $u_{j,\phi}(\omega) = [d\mu_{j,\phi}/d\mu_j](\omega)$ .

Of course, not every measure so constructed will be quasiinvariant. The idea that leads to an interesting class of quasiinvariant measures is to *scale* the probability distribution of the  $j$ th particle's position according to the *outcomes* for the previously chosen particle positions. This establishes a self-similar random process, where in the vicinity of accumulation points the ratio of probability density functions in Eq. (14) approaches the inverse of the Jacobian as  $j \rightarrow \infty$ . The resulting physical systems behave like an interacting gas of particles with one or more loci of condensation. However, our approach differs from the usual one in that our probability measures are constructed directly, rather than by means of an interaction Hamiltonian.

In general, if the positions of the particle coordinates  $\mathbf{x}_j(\omega)$ , or the successive difference coordinates  $\mathbf{y}_{j+1}(\omega) = \mathbf{x}_{j+1}(\omega) - \mathbf{x}_j(\omega)$ , distribute independently but non-identically—so that points can accumulate with non-zero probability—the resulting measure will not be quasiinvariant. However, Ismagilov did demonstrate quasiinvariance under diffeomorphisms of the measures resulting from a particular class of processes of this type, in one space dimension [20].

Sakuraba [37] showed that the quasiinvariant measures constructed by Goldin and Moschella from self-similar random processes, and the quasiinvariant measures of Ismagilov, are mutually singular.

### A. Example for $d = 1$

Let us illustrate with the examples based on Gaussian probability densities. Working first with  $d = 1$ , choose an initial point  $x_0$  from a nowhere vanishing probability density  $f_0$  on  $\mathbb{R}$ . For  $j = 1, 2, 3, \dots$ , let  $x_j = x_{j-1} + y_j$ , where the  $y_j$  are a sequence of deviation values. Choose the value  $y_1$  from a unit normal distribution  $g_1$ , with mean 0. Given the values  $y_1, \dots, y_j$ , choose  $y_{j+1}$  from a normal distribution with mean 0, and standard deviation  $\sigma_j = \kappa |y_j|$ , where  $\kappa > 0$  is a fixed *correlation parameter* independent of  $j$ . Small values of  $\kappa$  correspond to more tightly bound systems. Thus we have the conditional probability densities for the  $y_j$ ,

$$g_{j+1}^\kappa(y_{j+1} | y_j) = \frac{(2\pi)^{-\frac{1}{2}}}{\kappa |y_j|} \exp \left[ -\frac{1}{2\kappa^2} \left( \frac{y_{j+1}}{y_j} \right)^2 \right]. \quad (15)$$

For sufficiently small values of  $\kappa$ ,  $(y_j)$  converges to 0 (with probability one), while  $\sum_{j=1}^{\infty} |y_j| < \infty$ .

Let  $\text{Diff}_0^c(\mathbb{R})$  denote the stability subgroup of  $\text{Diff}^c(\mathbb{R})$ , consisting of the compactly supported diffeomorphisms of  $\mathbb{R}$  that leave the origin fixed. The measure on the space of sequences  $(y_j)$  resulting from the densities in Eq. (15) is then quasiinvariant under the action of elements of

$Diff_0^c(\mathbb{R})$ . We thus obtain the random sequence  $\omega = (x_k)$ , with  $x_k = x_0 + \sum_{j=1}^k y_j$ , and the corresponding random configuration  $\gamma = \{x_k\}$ .

Defining the terms  $u_{j,\phi}$  in Eq. (12) accordingly, we obtain  $u_j \rightarrow 1$  sufficiently rapidly to ensure convergence of the infinite product. More precisely, there exists a critical value  $\kappa_0$  such that if  $0 < \kappa < \kappa_0$ , sequences  $(x_j)$  converge to an accumulation point with probability one, while if  $\kappa_0 < \kappa$ , sequences diverge geometrically with probability one. In both cases, the associated measures on  $\Sigma_{\mathbb{R}}^\infty$  are quasiinvariant under compactly supported diffeomorphism of  $\mathbb{R}$  [32, 33, 34, 37]. The proofs make use of the strong law of large numbers.

The above is not tied essentially to the use of normal distributions; all that is really necessary for is the scaling property. Thus, for a whole class of models, there exists a critical value  $\kappa_0$  of the scaling parameter  $\kappa$ . For  $0 < \kappa < \kappa_0$ , the generalized configuration  $\{x_j\}$  has an accumulation point with probability one; we call this the *condensed phase*. For  $\kappa_0 < \kappa$ ,  $\{x_j\}$  has zero average density; we call this the *rarefied phase*. For each value of  $\kappa$  (except for the critical value itself), we have a bona fide unitary representation of  $Diff^c(\mathbb{R})$ , describing the associated quantum system.

## B. Generalization to $d > 1$

It was suggested earlier that a procedure similar to that suggested by Eq. (15) would work in  $d$  space dimensions,  $d > 1$ , to yield measures on the space of generalized configurations quasiinvariant under  $Diff^c(\mathbb{R}^d)$ ; with the conditional probability density for  $\mathbf{y}_{j+1}$  dependent on the preceding  $d$  outcomes  $(\mathbf{y}_{j-d+1}, \dots, \mathbf{y}_j)$  through the covariance matrix of a multivariate normal distribution [33, 34]. The generalization obtained by Sakuraba [37] achieves this, but also involves some new aspects.

Consider a random process where, at each stage,  $d$  vectors in  $\mathbb{R}^d$  are to be selected. Thus at each stage we are choosing a  $d \times d$  random matrix  $V$ , and it is appropriate to think of  $\omega \in [\mathbb{R}^d]^\infty$  as the sequence of square matrices  $([\mathbf{x}_1, \dots, \mathbf{x}_d], [\mathbf{x}_{d+1}, \dots, \mathbf{x}_{2d}], \dots)$ .

For the square matrix  $Y = [y_{ij}]$ , define the norm  $\|Y\| = [\sum_{i,j=1}^d y_{ij}^2]^{1/2}$ . Note that  $\|Y\|$  is a vector norm, not the operator norm of the matrix. For  $Y \in GL(d, \mathbb{R})$ , define the *condition number*  $k(Y) = \|Y\| \cdot \|Y^{-1}\|$ . We may write  $Y = P|Y|$ , where  $P$  is an orthogonal matrix and where  $|Y| = \sqrt{Y^t Y}$  is positive. Let  $\tau_1, \dots, \tau_d$  be eigenvalues of the matrix  $|Y|$ . Then  $\|Y\| = \|\tau_i\|$ , and

$$\|Y\| = [\sum_{i=1}^d \tau_i^2]^{1/2}, \quad \|Y^{-1}\| = [\sum_{j=1}^d \tau_j^{-2}]^{1/2},$$

$$k(Y) = [\sum_{i,j=1}^d (\tau_i/\tau_j)^2]^{1/2}. \quad (16)$$

Evidently  $k(Y)$  characterizes the amount of deformation under linear transformation by  $Y$ . If  $Y$  is not invertible,

then  $k(Y)$  is undefined (or infinite). Such matrices belong to measure zero sets in the constructions that follow.

We next construct a measure on  $[GL(d, \mathbb{R})]^\infty$  and thus on  $[\mathbb{R}^d]^\infty$ , quasiinvariant under  $Diff_0^c(\mathbb{R}^d)$ . Define the probability density function

$$f(Y) = C \exp \left\{ -\frac{1}{2\kappa^2} [\|Y\| k(Y)]^2 \right\} \quad (17)$$

on the set of  $d \times d$  matrices, where  $C$  is a normalization constant chosen so that  $\int f(Y) dY = 1$ ; here  $dY = d\mathbf{y}_1 \dots d\mathbf{y}_d$ . Let  $\mu^{(k)}$  be the probability measure defined by

$$d\mu^{(k)} = f(Y_1) \frac{f(Y_1^{-1}Y_2)}{|\det Y_1|^d} \dots \frac{f(Y_{k-1}^{-1}Y_k)}{|\det Y_{k-1}|^d} dY_1 \dots dY_k, \quad (18)$$

where  $d\mu^{(k)} = d\mu^{(k)}(Y_1, \dots, Y_k)$ . Then  $\mu^{(k)}$  is concentrated on  $[GL(d, \mathbb{R})]^k$ ; i.e., the set of sequences with one or more non-invertible matrices is of measure zero.

Then we again have a critical value  $\kappa_0$ . For  $\kappa < \kappa_0$ , the sequence  $(Y_j)$  of matrices—and thus the sequence of component vectors  $(\mathbf{y}_i)$ —converges to 0 with probability one; while for  $\kappa_0 < \kappa$ , it diverges with probability one. Furthermore, the projective limit measure  $\mu$  on  $[GL(d, \mathbb{R})]^\infty$  has the desired property of quasiinvariance under  $Diff_0^c(\mathbb{R}^d)$ . The presence of the condition number  $k(Y)$  in Eq. (17) is essential for the estimates required in demonstrating convergence of the infinite product in the resulting expression for the RN derivative. The proof here again uses the strong law of large numbers.

Eq. (17) can be generalized, replacing  $k(Y)$  by  $k(Y)^\alpha$  ( $\alpha \geq 1$ ), and replacing the Gaussian density by a more general probability density function.

Finally, we may begin with a matrix of positions  $X_0 = [\mathbf{x}_1, \dots, \mathbf{x}_d]$ , chosen from a nowhere vanishing probability density. Let  $\bar{\mathbf{x}}_0$  be the center of position of the  $d$  vectors comprising  $X_0$ . Now we may treat each new matrix  $Y_j$  as a set of deviations from the center of position of the preceding set of vectors  $X_{j-1}$ ; so that with obvious notation,  $X_j = \bar{\mathbf{x}}_{j-1} + Y_j$ . In this manner, we obtain a measure on  $[\mathbb{R}^d]^\infty$  quasiinvariant under  $Diff^c(\mathbb{R}^d)$ , that projects to a quasiinvariant measure on the space  $\Sigma_{\mathbb{R}^d}^\infty$  of generalized configurations.

More details about the preceding results may be found in the thesis of Sakuraba [37], and in forthcoming publications.

## VI. CONCLUSION

We believe the work summarized here strengthens the case for basing a theory of statistical physics in the manifold  $M$  on the configuration space  $\Sigma_M$  of countable subsets of  $M$ , endowed with the Vietoris topology. Measures obtained from random point processes in  $M$  project to measures on  $\Sigma_M$ , and when we consider self-similar random processes, we obtain measures quasiinvariant under the group of compactly-supported diffeomorphisms of  $M$ .



The problem of relating these measures to Hamiltonians on a classical phase space remains open.

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